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STRONG MODERATE DEVIATION THEOREMS FOR m-DEPENDENT RANDOM VARIABLES

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Strong Moderate Deviation Theorems for m-Dependent Random Variables†

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Abstract

Consider a stationary sequence $\{X_1, X_2, \ldots\}$ of m-dependent random variables. Let $S_n = \sum_{i=1}^{n} X_i$ be the partial sum. Under some moment conditions, in this paper we obtain asymptotic expression for the probability of moderate deviations, $P(S_n > x_n)$, where $x_n = O(\sqrt{\log(n)})$. This result extends some well known results obtained for independent and identically distributed sequences of random variables.

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1. Introduction. Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with mean 0 and variance 1. Let $S_n = \sum_{i=1}^{n} X_i$ be the n$^{\text{th}}$ partial sum. The theory of moderate deviations introduced by Rubin and Sethuraman (1965a) is concerned with obtaining asymptotic expression for

$$P(S_n > x_n) \tag{1.1}$$

where $x_n = O(\sqrt{\log(n)})$, under some moment conditions which are less restrictive than the assumption of finiteness of the moment generating function of X_1. In a subsequent paper Rubin and Sethuraman (1965b) showed that the asymptotic expression for (1.1) is useful to compare test statistics via Bayes risk efficiency. We shall call a result which gives the asymptotic expression for (1.1) a weak moderate deviation result. On the other hand a strong moderate deviation theorem gives an asymptotic expression to the probability of the event $\{S_n > x\}$ which is valid uniformly in the interval $-A \leq x \leq c\sqrt{\log(n)}$. In this paper we obtain strong moderate deviation theorem for the partial sums of a stationary sequence $\{X_n,\ n \geq 1\}$ of m- dependent random variables. Note that a sequence $\{X_n,\ n \geq 1\}$ is said to be m-dependent if $(X_1, \ldots, X_r)$ and $(X_s, X_{s+1}, \ldots)$ are independent whenever $s - r > m$. The sequence is said to be stationary if $(X_{i+1}, \ldots, X_{i+k})$ has the same distribution as $(X_{j+1}, \ldots, X_{j+k})$ for all $k \geq 1$ and $i \neq j$.

2. Main Results. In this section we establish the main theorem of this paper. Theorem 2.1 below obtains a strong moderate deviation theorem for partial sums of m-dependent sequence of random variables.

Theorem 2.1. *Consider a stationary sequence $\{X_n, n \geq 1\}$ of m-dependent random variables. Let $E(X_1) = 0$ and $\sigma^2 = Var(X_1) + 2\sum_{j=1}^{m} Cov(X_1, X_{1+j})$ be finite. Let $S_n = \sum_{i=1}^{n} X_i$. If $E|X_1|^p < \infty$ for some $p > c^2 + 2$, where $c > 0$, then*

$$P\left(\frac{S_n}{\sqrt{n}\sigma} > x\right) = [1 - \Phi(x)]\left[1 + o\left(\frac{1}{\log(n)}\right)\right] \tag{2.1}$$

uniformly in the region $-A \leq x \leq c\sqrt{\log(n)}$, where $A \geq 0$ is a constant and Φ denotes the distribution function of standard normal.

Vandemaele and Veraverbeke (1982) obtained strong moderate deviation theorems for L-statistics which are functions of independent and identically distributed sequences of random variables. A special case of their Theorem 1 yields the following Lemma 2.2. We will need Lemma 2.2 in the proof of Theorem 2.1.

Lemma 2.2. *Let $\{X_n, n \geq 1\}$ be sequence of i.i.d. random variables with mean zero and variance σ^2. Let $S_n = \sum_{i=1}^{n} X_i$ be the n^{th} partial sum. If $E|X_1|^p < \infty$ for some $p > c^2 + 2$, $(c > 0)$ then (2.1) holds uniformly in the region $-A \leq x \leq c\sqrt{\log(n)}$, where $A \geq 0$ is a constant.*

In the case $m = 0$, Theorem 2.1 yields Lemma 2.2 and in this sense our main result generalizes the result of Vandemaele and Veraverbeke (1982) to m-dependent random variables.

Proof of Theorem 2.1. We shall use the blocking technique used by Hoeffding and Robbins (1948) in proving central limit theorem for m-dependent random variables. Let $0 < \alpha < \min\{1, 1/p\}$ be fixed and $k = \max\{2m, [n^\alpha]\}$. We can write $n = k\nu + r$, where

$0 \leq r \leq k$. For n sufficiently large we can partition the n^{th} partial sum as follows:

$$
\begin{aligned}
S_n &= [X_1 + \ldots + X_{k-m}] + [X_{k-m+1} + \ldots + X_k] + [X_{k+1} + \ldots + X_{2k-m}] \\
&\quad + [X_{2k-m+1} + \ldots + X_{2k}] + \ldots + [X_{\nu k-m+1} + \ldots + X_{\nu k}] \\
&\quad + [X_{\nu k+1} + \ldots + X_n]. \\
&= U_{n1} + R_{n1} + U_{n2} + R_{n2} + \ldots + U_{n\nu} + R_{nu} + T_{nr} \\
&= [U_{n1} + \ldots + U_{n\nu}] + [R_{n1} + R_{n2} + \ldots + R_{n\nu}] + T_{nr} \\
&= U_n + R_n + T_{nr}. \qquad \text{(say)}.
\end{aligned}
\tag{2.2}
$$

Note that U_n is the sum of ν i.i.d. random variables and R_n is also the sum of ν i.i.d. random variables. Let $\delta_n = 1/(\log(n))^2$. It is easy to verify that the following important identity holds:

$$
\begin{aligned}
&P\left(\frac{U_n}{\sqrt{n}\sigma} > x + 2\delta_n\right) - P\left(\frac{R_n}{\sqrt{n}\sigma} < -\delta_n\right) - P\left(\frac{T_{nr}}{\sqrt{n}\sigma} < -\delta_n\right) \\
&\leq P\left(\frac{S_n}{\sqrt{n}\sigma} > x\right) \\
&\leq P\left(\frac{U_n}{\sqrt{n}\sigma} > x - 2\delta_n\right) + P\left(\frac{R_n}{\sqrt{n}\sigma} > \delta_n\right) + P\left(\frac{T_{nr}}{\sqrt{n}\sigma} > \delta_n\right).
\end{aligned}
\tag{2.3}
$$

Since $[1 - \Phi(x)]^{-1} = O((c\sqrt{\log(n)})n^{c^2/2})$ uniformly in $-A \leq x \leq c\sqrt{\log(n)}$, $(A > 0, c > 0)$, the proof of the theorem will be complete once we establish the following Lemma 2.3.

Lemma 2.3. *Let U_n, R_n and T_{nr} be as defined above. Then under the hypothesis of Theorem 2.1 we have the following:*

(A) $P\left(\frac{U_n}{\sqrt{n}\sigma} > x \pm 2\delta_n\right) = [1 = \Phi(x)]\left[1 + o\left(\frac{1}{\log(n)}\right)\right]$

uniformly in $-A \leq x \leq c\sqrt{\log(n)}$.

(B) $P\left(\left|\frac{R_n}{\sqrt{n}\sigma}\right| > \delta_n\right) = o\Big(\big(\log(n)\big)^{-3/2}\, n^{-c^2/2}\Big).$

(C) $P\left(|\frac{T_{nr}}{\sqrt{n}\sigma}| > \delta_n\right) = o\Big(\big(\log(n)\big)^{-3/2}\, n^{-c^2/2}\Big).$

Proof of (A). Note that $U_n' = \frac{U_n}{k}$ is the sum of ν i.i.d. random variables with mean zero and variance given by

$$M^2 = \frac{1}{k^2}[(k-m)\,Var(X_1) + 2\sum_{j=1}^{m}(k-m-j)\,Cov(X_1,\, X_{1+j})]. \tag{2.4}$$

Now,

$$\begin{aligned} P\left(\frac{U_n}{\sqrt{n}\sigma} > x\right) &= P\left(\frac{\sqrt{\nu}kM}{\sqrt{n}\sigma}\frac{U_n'}{M\sqrt{\nu}} > x\right) \\ &= P\left(\frac{U_n'}{M\sqrt{\nu}} > xc_n\right) \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} c_n = \frac{\sqrt{n}\sigma}{M\sqrt{\nu}k} &= \left[\frac{\sigma^2}{kM^2}\right]^{1/2}\left[\frac{n}{k\nu}\right]^{1/2} \\ &= \left[\frac{\sigma^2}{kM^2}\right]^{1/2}\left[1 + \frac{r}{k\nu}\right]^{1/2}. \end{aligned} \tag{2.6}$$

Note that

$$\begin{aligned} kM^2 &= \frac{1}{k}[(k-m)\,Var(X_1) + 2\sum_{j=1}^{m}(k-m-j)\,Cov(X_1, X_{1+j})] \\ &= \sigma^2 - \frac{1}{k}[mVar(X_1) + 2\sum_{j=1}^{m}(m+j)\mathrm{Cov}(X_1, X_{1+j})] \\ &= \sigma^2 + O(n^{-\alpha}) = \sigma^2[1 + O(n^{-\alpha})] \end{aligned} \tag{2.7}$$

since $k = O(n^{\alpha})$. Thus we get

$$\begin{aligned} c_n &= \left[1 + O(n^{-\alpha})\right]^{1/2}\left(1 + \frac{r}{k\nu}\right)^{1/2} \\ &= \left[1 + O(n^{-\beta})\right] \qquad \text{for some} \quad \beta > 0. \end{aligned} \tag{2.8}$$

Therefore we have shown that

$$P\left(\frac{U_n}{\sqrt{n}\sigma} > x\right) = P\left(\frac{U'_n}{M\sqrt{\nu}} > c_n x\right)$$

where $c_n = [1 + O(n^{-\beta})]$. Hence,

$$P\left(\frac{U_n}{\sqrt{n}\sigma} > x \pm \frac{2}{(\log(n))^2}\right) = P\left(\frac{U'_n}{M\sqrt{\nu}} > c_n\left[x \pm \frac{2}{(\log(n))^2}\right]\right)$$

(2.10)

$$= P\left(\frac{U'_n}{M\sqrt{\nu}} > x_n\right)$$

where

$$x_n = c_n\left[x \pm \frac{2}{(\log(n))^2}\right]$$

(2.11)

$$= [1 + O(n^{-\beta})]\left[x \pm \frac{2}{(\log(n))^2}\right].$$

Let δ be such that $\sqrt{p-2} > c + \delta$. Then applying Lemma 2.2 with c replaced by $c+\delta$ we get

(2.12) $$P\left(\frac{U'_n}{M\sqrt{\nu}} > x_n\right) = [1 - \Phi(x_n)]\left[1 + o\left(\frac{1}{\log(n)}\right)\right]$$

uniformly in the interval $-A \le x_n \le (c+\delta)\sqrt{\log(n)}$. Also using Lemma A1 of Vandermaele and Veraverbeke (1982) and (2.11) we get

(2.13) $$[1 - \Phi(x_n)] = [1 - \Phi(x)]\left[1 + o\left(\frac{1}{\log(n)}\right)\right].$$

Combining (2.10), (2.12) and (2.13) we have

$$P\left(\frac{U_n}{\sqrt{n}\sigma} > x \pm \frac{2}{(\log(n))^2}\right) = [1 - \Phi(x)]\left[1 + o\left(\frac{1}{\log(n)}\right)\right]$$

uniformly in $-A \le x \le c\sqrt{\log(n)}$. This completes the proof of (A).

Proof of (B). Note that R_n is the sum of ν i.i.d. random variables with mean zero and variance

$$\sigma_1^2 = mVar(X_1) + 2\sum_{j=1}^{m-1}(m-j)Cov(X_1, X_{1+j}). \tag{2.14}$$

Fix constants α and c_1 such that $\alpha < 1 - \frac{c^2}{p-2}$, $c_1^2 > \frac{c^2}{(1-\alpha)}$ and $p > 2 + c_1^2$. Applying Theorem 1 of Rubin and Sethuraman (1965a) we get for sufficiently large n,

$$\begin{aligned} P\left(\left|\frac{R_n}{\sqrt{n}\sigma}\right| > \delta_n\right) &= P\left(\left|\frac{R_n}{\sqrt{\nu}}\right| > \frac{\sigma\sqrt{n}}{\sqrt{\nu}(\log(n))^2}\right) \\ &\leq P\left(\left|\frac{R_n}{\sqrt{\nu}}\right| > \sigma_1 c_1\sqrt{\frac{\log(\nu)}{\nu}}\right) \\ &\sim \frac{\nu^{-c_1^2/2}}{c_1\sqrt{2\pi\log(\nu)}} \end{aligned} \tag{2.15}$$

since $p > c_1^2 + 2$. Using the fact $\nu = O(n^{1-\alpha})$ and $(1-\alpha)c_1^2 > c^2$ and (2.15), we get that

$$P\left(\left|\frac{R_n}{\sqrt{n}\sigma}\right| > \delta_n\right) = o\left((\log(n))^{-3/2}n^{-c^2/2}\right).$$

This proves (B).

Proof of (C). Applying Chebyshev's inequality we get

$$\begin{aligned} P\left(\left|\frac{T_{nr}}{\sqrt{n}\sigma}\right| > \delta_n\right) &\leq \frac{E|T_{n,r}|^p}{(\sigma\sqrt{n}\delta_n)^p} \\ &\leq \text{const.}\frac{k^p}{(\sigma\sqrt{n}\delta_n)^p}. \end{aligned} \tag{2.16}$$

Note that $k = O(n^\alpha)$ and hence

$$n^{c^2/2}(\log(n))^{3/2}\frac{k^p}{(\sqrt{n}\delta_n)^p} = \frac{n^{c^2/2}n^{p\alpha}}{n^{p/2}}(\log(n))^{2p+3/2}$$

$$(2.17) \qquad = n^{-\frac{1}{2}(p-2p\alpha-c^2)}(\log(n))^{2p+3/2}$$

$$\to 0 \qquad \text{as } n \to \infty,$$

since $0 < \alpha < \frac{1}{p}$ and $p > c^2 + 2$. The proof of (C) now follows combining (2.16) and (2.17).

Remark 2.4. Theorem 2.1 suggests that it is possible to obtain strong moderate deviation theorems for L-statistics and U-statistics which are functions of m-dependent sequences of random variables in the same spirit as Vandemaele and Veraverbeke(1982) and Ghosh(1974).

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